8-6 Solving Rational Equations and Inequalities

Main Ideas

- Solve rational equations.
- Solve rational inequalities.

New Vocabulary

rational equation
rational inequality

GET READY for the Lesson

A music download service advertises downloads for \$1 per song. The service also charges a monthly access fee of \$15. If a customer downloads x songs in one month, the bill in dollars will be $15 + x$. The actual cost per song is $\frac{15 + x}{x}$. To find how many songs a person would need to download to make the actual cost per song \$1.25, you would need to solve the equation $\frac{15 + x}{x} = 1.25$.

Solve Rational Equations The equation $\frac{15 + x}{x} = 6$ is an example of a rational equation. In general, any equation that contains one or more rational expressions is called a **rational equation**.

Rational equations are easier to solve if the fractions are eliminated. You can eliminate the fractions by multiplying each side of the equation by the least common denominator (LCD). Remember that when you multiply each side by the LCD, each term on each side must be multiplied by the LCD.

EXAMPLE 1 Solve a Rational Equation

Solve $\frac{9}{28} + \frac{3}{z + 2} = \frac{3}{4}$. Check your solution.

The LCD for the terms is $28(z + 2)$.

$$\frac{9}{28} + \frac{3}{z + 2} = \frac{3}{4} \quad \text{Original equation}$$

$$28(z + 2)\left(\frac{9}{28} + \frac{3}{z + 2}\right) = 28(z + 2)\left(\frac{3}{4}\right) \quad \text{Multiply each side by } 28(z + 2).$$

$$\overset{1}{\cancel{28}}(z + 2)\left(\frac{9}{\underset{1}{\cancel{28}}}\right) + 28\overset{1}{\cancel{(z + 2)}}\left(\frac{3}{\underset{1}{\cancel{z + 2}}}\right) = \overset{7}{\cancel{28}}(z + 2)\left(\frac{3}{\underset{1}{\cancel{4}}}\right) \quad \text{Distributive Property}$$

$$(9z + 18) + 84 = 21z + 42 \quad \text{Simplify.}$$

$$9z + 102 = 21z + 42 \quad \text{Simplify.}$$

$$60 = 12z \quad \text{Subtract } 9z \text{ and } 42 \text{ from each side.}$$

$$5 = z \quad \text{Divide each side by 12.}$$

(continued on the next page)

CHECK $\frac{9}{28} + \frac{3}{z+2} = \frac{3}{4}$ Original equation

$\frac{9}{28} + \frac{3}{5+2} \stackrel{?}{=} \frac{3}{4}$ $z = 5$

$\frac{9}{28} + \frac{3}{7} \stackrel{?}{=} \frac{3}{4}$ Simplify.

$\frac{9}{28} + \frac{12}{28} \stackrel{?}{=} \frac{3}{4}$ Simplify.

$\frac{3}{4} = \frac{3}{4}$ ✓ The solution is correct.

CHECK Your Progress

Solve each equation. Check your solution.

1A. $\frac{5}{6} + \frac{2}{x-6} = \frac{1}{2}$

1B. $\frac{7}{12} + \frac{9}{x-4} = \frac{55}{48}$

When solving a rational equation, any possible solution that results in a zero in the denominator must be excluded from your list of solutions.

EXAMPLE 2 Elimination of a Possible Solution

Solve $r + \frac{r^2-5}{r^2-1} = \frac{r^2+r+2}{r+1}$**. Check your solution.**

$r + \frac{r^2-5}{r^2-1} = \frac{r^2+r+2}{r+1}$	Original equation; the LCD is $(r^2 - 1)$.
$(r^2-1)\left(r + \frac{r^2-5}{r^2-1}\right) = (r^2-1)\left(\frac{r^2+r+2}{r+1}\right)$	Multiply each side by the LCD.
$(r^2-1)r + \cancel{(r^2-1)}^{1}\left(\frac{r^2-5}{\cancel{r^2-1}_{1}}\right) = \cancel{(r^2-1)}\left(\frac{r^2+r+2}{\cancel{r+1}_{1}}\right)$	Distributive Property
$(r^3 - r) + (r^2 - 5) = (r-1)(r^2+r+2)$	Simplify.
$r^3 + r^2 - r - 5 = r^3 + r - 2$	Simplify.
$r^2 - 2r - 3 = 0$	Subtract $(r^3 + r - 2)$ from each side.
$(r-3)(r+1) = 0$	Factor.
$r - 3 = 0$ or $r + 1 = 0$	Zero Product Property
$r = 3$ $\quad r = -1$	

CHECK $r + \frac{r^2-5}{r^2-1} = \frac{r^2+r+2}{r+1}$

$3 + \frac{3^2-5}{3^2-1} \stackrel{?}{=} \frac{3^2+3+2}{3+1}$

$3 + \frac{4}{8} \stackrel{?}{=} \frac{14}{4}$

$\frac{7}{2} = \frac{7}{2}$ ✓

$r + \frac{r^2-5}{r^2-1} = \frac{r^2+r+2}{r+1}$

$-1 + \frac{(-1)^2-5}{(-1)^2-1} \stackrel{?}{=} \frac{(-1)^2+(-1)+2}{-1+1}$

$-1 + \frac{-4}{0} \stackrel{?}{=} \frac{2}{0}$

Since $r = -1$ results in a zero in the denominator, eliminate -1 from the list of solutions. The solution is 3.

Study Tip

Extraneous Solutions

Multiplying each side of an equation by the LCD of rational expressions can yield results that are not solutions of the original equation. These solutions are called *extraneous solutions*.

CHECK Your Progress Solve each equation. Check your solution.

2A. $\frac{2}{r+1} - \frac{1}{r-1} = \frac{-2}{r^2-1}$ **2B.** $\frac{7n}{3n+3} - \frac{5}{4n-4} = \frac{3n}{2n+2}$

Real-World Link

The Loetschberg Tunnel is 21 miles long. It was created for train travel and cut travel time between the locations in half.

Source: usatoday.com

Real-World EXAMPLE

3 **TUNNELS** **The Loetschberg tunnel was built to connect Bern, Switzerland, with the ski resorts in the southern Swiss Alps. The Swiss used one company that started at the north end and another company that started at the south end. Suppose the company at the north end could drill the entire tunnel in 22.2 years and the south company could do it in 21.8 years. How long would it have taken the two companies to drill the tunnel?**

In 1 year, the north company could complete $\frac{1}{22.2}$ of the tunnel.

In 2 years, the north company could complete $\frac{1}{22.2} \cdot 2$ or $\frac{2}{22.2}$ of the tunnel.

In t years, the north company could complete $\frac{1}{22.2} \cdot t$ or $\frac{t}{22.2}$ of the tunnel.

Likewise, in t years, the south company could complete $\frac{1}{21.8} \cdot t$ or $\frac{t}{21.8}$ of the tunnel.

Together, they completed the whole tunnel.

Part completed by the north company	plus	part completed by the south company	equals	entire tunnel.
$\frac{t}{22.2}$	$+$	$\frac{t}{21.8}$	$=$	1

$\frac{t}{22.2} + \frac{t}{21.8} = 1$ Original equation

$483.96\left(\frac{t}{22.2} + \frac{t}{21.8}\right) = 483.96(1)$ Multiply each side by 483.96.

$21.8t + 22.2t = 483.96$ Simplify.

$44t = 483.96$ Simplify.

$t \approx 11$ Divide each side by 44.

It would have taken about 11 years to build the tunnel.

This answer is reasonable. Working alone, either company could have drilled the tunnel in about 22 years. Working together, they must be able to do it in about half that time.

CHECK Your Progress

3. WORK Breanne and Owen paint houses together. If Breanne can paint a particular house in 6 days and Owen can paint the same house in 5 days, how long would it take the two of them if they work together?

Online Personal Tutor at algebra2.com

Rate problems frequently involve rational equations.

Real-World EXAMPLE

4 **NAVIGATION** **The speed of the current in the Puget sound is 5 miles per hour. A barge travels 26 miles with the current and returns in $10\frac{2}{3}$ hours. What is the speed of the barge in still water?**

Words The formula that relates distance, time, and rate is $d = rt$ or $\frac{d}{r} = t$.

Variables Let r = the speed of the barge in still water. Then the speed of the barge with the current is $r + 5$, and the speed of the barge against current is $r - 5$.

	Time going with the current	plus	time going against the current	equals	total time.
Equation	$\frac{26}{r+5}$	$+$	$\frac{26}{r-5}$	$=$	$10\frac{2}{3}$

$$\frac{26}{r+5} + \frac{26}{r-5} = 10\frac{2}{3}$$ Original equation

$$3(r^2 - 25)\left(\frac{26}{r+5} + \frac{26}{r-5}\right) = 3(r^2 - 25)10\frac{2}{3}$$ Multiply each side by $3(r^2 - 25)$.

$$3(r^2 - 25)\left(\frac{26}{r+5}\right) + 3(r^2 - 25)\left(\frac{26}{r-5}\right) = 3(r^2 - 25)\left(\frac{32}{3}\right)$$ Distributive Property

$$(78r - 390) + (78r + 390) = 32r^2 - 800$$ Simplify.

$$156r = 32r^2 - 800$$ Simplify.

$$0 = 32r^2 - 156r - 800$$ Subtract $156r$ from each side.

$$0 = 8r^2 - 39r - 200$$ Divide each side by 4.

Use the Quadratic Formula to solve for r.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Quadratic Formula

$$r = \frac{-(-39) \pm \sqrt{(-39)^2 - 4(8)(-200)}}{2(8)}$$ $x = r$, $a = 8$, $b = -39$, and $c = -200$

$$r = \frac{39 \pm \sqrt{7921}}{16}$$ Simplify.

$$r = \frac{39 \pm 89}{16}$$ Simplify.

$$r = 8 \text{ or } -3.125$$ Simplify.

Since speed must be positive, it is 8 miles per hour. Is this answer reasonable?

Study Tip

Look Back
To review the **Quadratic Formula**, see Lesson 5-6.

CHECK Your Progress

4. SWIMMING The speed of the current in a body of water is 1 mile per hour. Juan swims 2 miles against the current and 2 miles with the current in a total time of $2\frac{2}{3}$ hours. How fast can Juan swim in still water?

Concepts in Motion
Interactive Lab
algebra2.com

Solve Rational Inequalities Inequalities that contain one or more rational expressions are called **rational inequalities**. To solve rational inequalities, complete the following steps.

Step 1 State the excluded values.

Step 2 Solve the related equation.

Step 3 Use the values determined in Steps 1 and 2 to divide a number line into intervals. Test a value in each interval to determine which intervals contain values that satisfy the original inequality.

EXAMPLE 5 Solve a Rational Inequality

5 Solve $\frac{1}{4a} + \frac{5}{8a} > \frac{1}{2}$.

Step 1 Values that make a denominator equal to 0 are excluded from the domain. For this inequality, the excluded value is 0.

Step 2 Solve the related equation.

$$\frac{1}{4a} + \frac{5}{8a} = \frac{1}{2}$$ Related equation

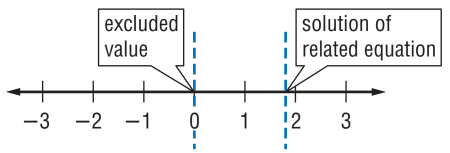
$$8a\left(\frac{1}{4a} + \frac{5}{8a}\right) = 8a\left(\frac{1}{2}\right)$$ Multiply each side by $8a$.

$$2 + 5 = 4a$$ Simplify.

$$7 = 4a$$ Add.

$$1\frac{3}{4} = a$$ Divide each side by 4.

Step 3 Draw vertical lines at the excluded value and at the solution to separate the number line into intervals.



Now test a sample value in each interval to determine if the values in the interval satisfy the inequality.

Test $a = -1$.	Test $a = 1$.	Test $a = 2$.
$\frac{1}{4(-1)} + \frac{5}{8(-1)} \stackrel{?}{>} \frac{1}{2}$	$\frac{1}{4(1)} + \frac{5}{8(1)} \stackrel{?}{>} \frac{1}{2}$	$\frac{1}{4(2)} + \frac{5}{8(2)} \stackrel{?}{>} \frac{1}{2}$
$-\frac{1}{4} - \frac{5}{8} \stackrel{?}{>} \frac{1}{2}$	$\frac{1}{4} + \frac{5}{8} \stackrel{?}{>} \frac{1}{2}$	$\frac{1}{8} + \frac{5}{16} \stackrel{?}{>} \frac{1}{2}$
$-\frac{7}{8} \ngtr \frac{1}{2}$	$\frac{7}{8} > \frac{1}{2}$ ✓	$\frac{7}{16} \ngtr \frac{1}{2}$
$a < 0$ is *not* a solution.	$0 < a < 1\frac{3}{4}$ is a solution.	$a > 1\frac{3}{4}$ is *not* a solution.

The solution is $0 < a < 1\frac{3}{4}$.

CHECK Your Progress

Solve each inequality.

5A. $\frac{1}{3b} - \frac{2}{5b} < \frac{1}{15}$

5B. $1 + \frac{5}{x-1} \leq \frac{7}{6}$

CHECK Your Understanding

Example 1 (pp. 479–480)

Solve each equation. Check your solutions.

1. $\frac{2}{d} + \frac{1}{4} = \frac{11}{12}$

2. $t + \frac{12}{t} - 8 = 0$

3. $\frac{1}{x-1} + \frac{2}{x} = 0$

4. $\frac{12}{v^2 - 16} - \frac{24}{v-4} = 3$

Example 2 (p. 480)

5. $\frac{w}{w-1} + w = \frac{4w-3}{w-1}$

6. $\frac{4n^2}{n^2 - 9} - \frac{2n}{n+3} = \frac{3}{n-3}$

Examples 3, 4 (pp. 481, 482)

7. **WORK** A worker can powerwash a wall of a certain size in 5 hours. Another worker can do the same job in 4 hours. If the workers work together, how long would it take to do the job? Determine whether your answer is reasonable.

Example 5 (p. 483)

Solve each inequality.

8. $\frac{4}{c+2} > 1$

9. $\frac{1}{3v} + \frac{1}{4v} < \frac{1}{2}$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
10–15	1
16, 17	2
18–21	5
22, 23	3, 4

Solve each equation or inequality. Check your solutions.

10. $\frac{y}{y+1} = \frac{2}{3}$

11. $\frac{p}{p-2} = \frac{2}{5}$

12. $s + 5 = \frac{6}{s}$

13. $a + 1 = \frac{6}{a}$

14. $\frac{9}{t-3} = \frac{t-4}{t-3} + \frac{1}{4}$

15. $\frac{5}{x+1} - \frac{1}{3} = \frac{x+2}{x+1}$

16. $\frac{2}{y+2} - \frac{y}{2-y} = \frac{y^2+4}{y^2-4}$

17. $\frac{1}{d+4} = \frac{2}{d^2+3d-4} - \frac{1}{1-d}$

18. $\frac{7}{a+1} > 7$

19. $\frac{10}{m+1} > 5$

20. $5 + \frac{1}{t} > \frac{16}{t}$

21. $7 - \frac{2}{b} < \frac{5}{b}$

22. **NUMBER THEORY** The ratio of 16 more than a number to 12 less than that number is 1 to 3. What is the number?

23. **NUMBER THEORY** The sum of a number and 8 times its reciprocal is 6. Find the number(s).

Solve each equation or inequality. Check your solutions.

24. $\frac{b-4}{b-2} = \frac{b-2}{b+2} + \frac{1}{b-2}$

25. $\frac{1}{n-2} = \frac{2n+1}{n^2+2n-8} + \frac{2}{n+4}$

26. $\frac{2q}{2q+3} - \frac{2q}{2q-3} = 1$

27. $\frac{4}{z-2} - \frac{z+6}{z+1} = 1$

28. $\frac{2}{3y} + \frac{5}{6y} > \frac{3}{4}$

29. $\frac{1}{2p} + \frac{3}{4p} < \frac{1}{2}$

30. **ACTIVITIES** The band has 30 more members than the school chorale. If each group had 10 more members, the ratio of their membership would be 3:2. How many members are in each group?

Real-World Career

Chemist

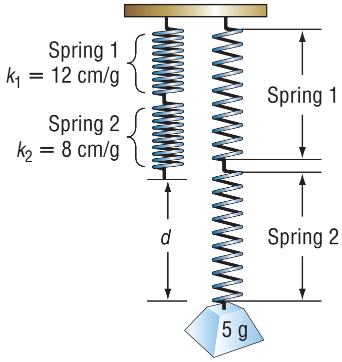
Many chemists work for manufacturers developing products or doing quality control to ensure the products meet industry and government standards.

Math Online

For more information, go to algebra2.com.

PHYSICS For Exercises 31 and 32, use the following information.

The distance a spring stretches is related to the mass attached to the spring. This is represented by $d = km$, where d is the distance, m is the mass, and k is the spring constant. When two springs with spring constants k_1 and k_2 are attached in a series, the resulting spring constant k is found by the equation $\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$.



31. If one spring with constant of 12 centimeters per gram is attached in a series with another spring with constant of 8 centimeters per gram, find the resultant spring constant.

32. If a 5-gram object is hung from the series of springs, how far will the springs stretch? Is this answer reasonable in this context?

33. CYCLING On a particular day, the wind added 3 kilometers per hour to Alfonso's rate when he was cycling with the wind and subtracted 3 kilometers per hour from his rate on his return trip. Alfonso found that in the same amount of time he could cycle 36 kilometers with the wind, he could go only 24 kilometers against the wind. What is his normal bicycling speed with no wind? Determine whether your answer is reasonable.

34. CHEMISTRY Kiara adds an 80% acid solution to 5 milliliters of solution that is 20% acid. The function that represents the percent of acid in the resulting solution is $f(x) = \frac{5(0.20) + x(0.80)}{5 + x}$, where x is the amount of 80% solution added. How much 80% solution should be added to create a solution that is 50% acid?

35. NUMBER THEORY The ratio of 3 more than a number to the square of 1 more than that number is less than 1. Find the numbers which satisfy this statement.

EXTRA PRACTICE

See pages 909, 933.

Math Online

Self-Check Quiz at algebra2.com

STATISTICS For Exercises 36 and 37, use the following information.

A number x is the *harmonic mean* of y and z if $\frac{1}{x}$ is the average of $\frac{1}{y}$ and $\frac{1}{z}$.

36. Eight is the harmonic mean of 20 and what number?

37. What is the harmonic mean of 5 and 8?

H.O.T. Problems

38. OPEN ENDED Write a rational equation that can be solved by first multiplying each side by $5(a + 2)$.

39. FIND THE ERROR Jeff and Dustin are solving $2 - \frac{3}{a} = \frac{2}{3}$. Who is correct? Explain your reasoning.

Jeff

$2 - \frac{3}{a} = \frac{2}{3}$

$6a - 9 = 2a$

$4a = 9$

$a = 2.25$

Dustin

$2 - \frac{3}{a} = \frac{2}{3}$

$2 - 9 = 2a$

$-7 = 2a$

$-3.5 = a$

40. CHALLENGE Solve for a if $\frac{1}{a} - \frac{1}{b} = c$.

41. *Writing in Math* Use the information about music downloads on page 479 to explain how rational equations are used to solve problems involving unit price. Include an explanation of why the actual price per download could never be $1.00.

STANDARDIZED TEST PRACTICE

42. **ACT/SAT** Amanda wanted to determine the average of her 6 test scores. She added the scores correctly to get T, but divided by 7 instead of 6. The result was 12 less than her actual average. Which equation could be used to determine the value of T?

A $6T + 12 = 7T$

B $\frac{T}{7} = \frac{T-12}{6}$

C $\frac{T}{7} + 12 = \frac{T}{6}$

D $\frac{T}{6} = \frac{T-12}{7}$

43. **REVIEW**
What is $\frac{10a^{-3}}{29b^4} \div \frac{5a^{-5}}{16b^{-7}}$?

F $\frac{25b^3}{232a^8}$

G $\frac{25}{232a^2b^3}$

H $\frac{32b^3}{29a^8}$

J $\frac{32a^2}{29b^{11}}$

Spiral Review

Identify the type of function represented by each equation. Then graph the equation. (Lesson 8-5)

44. $y = 2x^2 + 1$

45. $y = 2\sqrt{x}$

46. $y = 0.8x$

47. If y varies inversely as x and $y = 24$ when $x = 9$, find y when $x = 6$. (Lesson 8-4)

Solve each inequality. (Lesson 5-8)

48. $(x + 11)(x - 3) > 0$

49. $x^2 - 4x \leq 0$

50. $2b^2 - b < 6$

Find each product, if possible. (Lesson 4-3)

51. $\begin{bmatrix} 3 & -5 \\ 2 & 7 \end{bmatrix} \cdot \begin{bmatrix} 5 & 1 & -3 \\ 8 & -4 & 9 \end{bmatrix}$

52. $\begin{bmatrix} 4 & -1 & 6 \\ 1 & 5 & -8 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 9 & -6 \end{bmatrix}$

53. **HEALTH** The prediction equation $y = 205 - 0.5x$ relates a person's maximum heart rate for exercise y and age x. Use the equation to find the maximum heart rate for an 18-year-old. (Lesson 2-5)

Determine the value of r so that a line through the points with the given coordinates has the given slope. (Lesson 2-3)

54. $(r, 2), (4, -6)$; slope $= -\frac{8}{3}$

55. $(r, 6), (8, 4)$; slope $= \frac{1}{2}$

56. Evaluate $[(-7 + 4) \times 5 - 2] \div 6$. (Lesson 1-1)